Spin 1/2 quasinormal mode frequencies in Reissner-Nördstrom-AdS spacetime

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Abstract

We investigate quasinormal mode frequencies ω_n of the Dirac equation in a Reissner-Nördstrom-AdS$_D$ background in space-time dimension $D > 3$, in the black brane (large black hole) limit. By asymptotic, we mean large overtone number n with everything else held fixed.

1 Introduction

Quasinormal modes of black holes have been extensively studied [1], but with the introduction of gauge-gravity duality, asymptotically AdS black holes have seen increased interest. To our knowledge, there exists no analytic treatment of Ressiner-Nördstrom-AdS$_D$, except numerically [3], and in the low-dimensional case [4].

Here, following the treatment in Arnold [5] of the Schwartzschild case using WKB-like methods, we obtain analytic results for quasinormal mode frequencies ω_n in the limit of large overtone number (i.e. larger ω_n). In this paper, we analyze the quasinormal modes ω_n of a Dirac (i.e. spin $\frac{1}{2}$) particle in AdS-RN$_D$ for $D > 2 + 1$. Quasinormal modes are evenly spaced in the limit of large n, with asymptotic expansion

$$\omega_n = n\Delta\omega + A\ln n + B + \cdots$$

where other quantities (mass m, charge q, and momentum parallel to the horizon k_Ω) are held fixed [6]. We present an asymptotic formula for ω_n through $O(n^0)$, and find a more general formula that accounts for mass dependence.

1.1 Metric

The RNAdS$_D$ black-brane metric is written

$$ds^2 = \frac{L^2}{z^2}\left[-f dt^2 + d\vec{x}^2 + f^{-1} dz^2\right] \tag{1}$$
$$= -F dt^2 + r^2 \frac{d\vec{x}^2}{L^2} + F^{-1} dr^2, \tag{2}$$

where L is AdS radius,

$$z = L^2 r$$

We follow the conventions of [5] for f and F.

$$f \equiv 1 + \frac{z^2}{L^2} - z^d + q^2 z^{2d-2}$$

Since here we are considering the black brane limit (i.e. a very large black hole), we suppress the $\frac{z^2}{L^2}$ term, and use

$$f = 1 - z^d + q^2 z^{2d-2}$$

2 Derivation

2.1 The Dirac equation

The Dirac equations reads

$$(\not{D} - m)\,\Psi = 0$$

where

$$\not{D} = \Gamma^M D_M$$
$$\{\Gamma^M, \Gamma^N\} = 2g^{MN}$$

As in [7], we rescale

$$\psi \equiv (-g g^{zz})^{1/4} \Psi$$

Plugging in the metric, we have

$$(-f\partial_z + i\gamma^5\gamma^0\omega - i f^{1/2}\gamma^5 \boldsymbol{\gamma}\cdot\boldsymbol{k} + \bar{m}\frac{f^{1/2}}{z}\gamma^5)\psi = 0$$

where $\bar{m} = mL$

Using the Pauli matrices τ_1, τ_2, τ_3, we can write the naive large-ω limit as

$$(-f\partial_z + i\omega\tau_3)\psi = 0$$

with solutions

$$\psi = e^{-\omega r_* \tau_3}\eta$$

where η is a constant Dirac spinor, and r_* is the tortoise coordinate

$$r_* = \int_z^\infty \frac{dz}{f}$$

2.2 The Method

The naive solution

$$\psi = e^{-\omega r_* \tau_3}\eta = e^{-i\omega r_*}\eta^{(+)} + e^{i\omega r_*}\eta^{(-)}$$

has components that are positive and negative exponential in ωr_*. This means that we have to be careful about the range of applicability of this solution—we don't want one component to get lost in the potentially exponentially larger error in the other.

We resolve this by following this (WKB) solution along the Stokes lines, defined by $\Im(\omega r_*) = 0$. The WKB solution is not valid very close to $z = 0$ or $z = \infty$, so we separately solve the Dirac equation in these regions. We let $\eta_+^{(\pm)}$ be the components of the Dirac spinor along the positive Stokes line, and $\eta_-^{(\pm)}$ negative.

2.3 Behavior near the singularity

We find solutions near the singularity for both Stokes lines in the large z, large ω regime. We match them by dragging our solution from the positive Stokes line to the negative one, through an angle of $\frac{\pi}{d}$.

$$f = 1 + z^2 - \bar{m} z^d + q^2 z^{2(d-1)} \tag{3}$$
$$\simeq z^{2(d-1)} \tag{4}$$
$$f^{1/2} \simeq q z^{d-1} \tag{5}$$
$$\partial_{r_*}\eta \simeq z^{d-1} e^{2i\omega r_* \tau_3} \left(k\tau_1 + \frac{\bar{m}}{z}\tau_2 \right) \eta \qquad \eta(z) = \eta^{(0)} + \eta^{(1)}(z) \tag{6}$$
$$\eta^{(1)} \simeq \int_\infty^{r_*} dr_* z^{d-1} e^{2i\omega r_* \tau_3} \left(k\tau_1 + \frac{\bar{m}}{z}\tau_2 \right) \eta^{(0)} \tag{7}$$
$$\simeq \int_\infty^{r_*} dr_* \left[q(2d-1) r_* \right]^{\frac{1}{3-2d}} e^{2i\omega r_* \tau_3} \left(k\tau_1 + \frac{\bar{m}}{z}\tau_2 \right) \eta^{(0)} \tag{8}$$
$$= -q \left[q(2d-1) \right]^{\frac{d-1}{3-2d}} \int_\infty^{r_*} dr_* r_*^{\frac{d-1}{3-2d}} e^{-r_*(-2i\omega\tau_3)} \left(k\tau_1 + \frac{\bar{m}}{z}\tau_2 \right) \eta^{(0)} \tag{9}$$
$$\tag{10}$$

Letting

$$a = \frac{d-2}{2d-1} \tag{11}$$

$$\begin{aligned}
\eta^{(1)} &= -q^a (2d-1)^{a-1} \int_\infty^{r_*} dr_* r_*^{a-1} e^{-r_*(-2i\omega\tau_3)} \left(k\tau_1 + \frac{\bar{m}}{z}\tau_2 \right) \eta^{(0)} \\
&= -q^a (2d-1)^{a-1} \int_\infty^{r_*} dr_* r_*^{a-1} e^{-r_*(-2i\omega\tau_3)} \left(k\tau_1 + \bar{m} \left[q(2d-1) r_* \right]^{\frac{1}{2d-1}} \tau_2 \right) \eta^{(0)} \\
&= -q^a (2d-1)^{a-1} \frac{\Gamma(a, -2i\omega r_* \tau_3)}{(-2i\omega\tau_3)^a} k\tau_1 \eta^{(0)} - q^{1-a} (2d-1)^{-a} \frac{\Gamma(1-a, -2i\omega r_* \tau_3)}{(-2i\omega\tau_3)^a} \bar{m}\tau_2 \eta^{(0)} \\
&= -q^a (2d-1)^{a-1} \left[\frac{\Gamma(a, -2i\omega r_*)}{(-2i\omega)^a} k\tau_1 \eta^{(0)-} + \frac{\Gamma(a, 2i\omega r_*)}{(-i\omega)^a} k\tau_1 \eta^{(0)+} \right] \\
&\qquad - q^{1-a} (2d-1)^{-a} \left[\frac{\Gamma(1-a, -2i\omega r_*)}{(-2i\omega)^a} \bar{m}\tau_2 \eta^{(0)(-)} + \frac{\Gamma(1-a, 2i\omega r_*)}{(2i\omega)^a} \bar{m}\tau_2 \eta^{(0)(+)} \right]
\end{aligned}$$

To match with our WKB solutions, we take the asymptotic expansion of Γ,

$$\Gamma(\alpha, z) \simeq z^{\alpha-1} e^{-z} \qquad (|z| \to \infty \text{ with } |\arg z| < \frac{3\pi}{2}) \tag{12}$$

The domain restriction arises due to a branch cut of the incomplete Γ function in the z plane. In order to meet it, when we're on the negative Stokes line, we must use the monodromy relation:

$$\Gamma(\alpha, e^{i2\pi n} z) = [1 - e^{i2\pi n\alpha}]\Gamma(a) + e^{i2\pi n\alpha}\Gamma(\alpha, z)$$

This gives us:

$$\eta_- = \eta_+ + A\tau_1\eta_+^{(+)} - B\tau_2\eta_+^{(+)}$$

where

$$A = -q^a k e^{-i\pi a}(2d-1)^{a-1}\frac{\Gamma(a)(1-e^{2i\pi a})}{(-2i\omega)^a} \tag{13}$$

$$B = -q^a \bar{m} e^{i\pi a}(2d-1)^{-a}\frac{\Gamma(a)(1-e^{2i\pi(1-a)})}{(-2i\omega)^{(1-a)}} \tag{14}$$

$$\tag{15}$$

2.4 Behavior near the boundary

As in [6], we can just take $f \sim 1$, so

$$\psi \propto \sqrt{\pi\Omega z}\begin{pmatrix} J_{\bar{m}-\frac{1}{2}}(\Omega z) + i\frac{\Omega}{\Omega-k}J_{\bar{m}+\frac{1}{2}}(\Omega z) \\ \frac{\Omega}{\Omega-k}J_{\bar{m}+\frac{1}{2}}(\Omega z) + iJ_{\bar{m}-\frac{1}{2}}(\Omega z) \end{pmatrix}$$

In our limit, this reduces to

$$\psi \propto e^{i\omega r_* \tau_3}\begin{pmatrix} -ie^{-i\bar{m}\pi/2}e^{i\omega r_{*0}} \\ e^{i\bar{m}\pi/2}e^{-i\omega r_{*0}} \end{pmatrix}$$

Where we've taken $z \simeq r_* + r_{*0}$, where

$$r_{*0} = \int_0^\infty \frac{dz}{f}$$

2.5 Putting it together

$$0 = \eta_+^{(-)} + (\tau_1 A + \tau_2 B)\,\eta_+^{(+)} \tag{16}$$

$$1 = (A+B)ie^{i\bar{m}\pi}e^{2i\omega r_{*0}} \tag{17}$$

It is straightforward to solve this for ω up to $O(1)$.

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